Worcester County Mathematics League

Varsity Meet 3 January 24, 2018

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS



Varsity Meet 3 - January 24, 2018

ANSWER KEY

Round 1:

- 1. 20 (Hudson)
- 2. 8 (Notre Dame)
- 3. 2√29 (Algonquin)

Round 2:

- 1. 17 (Shepherd Hill)
- 2. -12 (Algonquin)
- 3. 140 (Worcester Academy)

Round 3:

- 1. 5 (Leicester)
- 2. $\pm \frac{\sqrt{6}}{2}$ (Bromfield)
- 3. -8, 3 or 3, -8 (Doherty)

Round 4:

- 1. 585,000 (Westboro)
- 2. 17 (Mass Academy)
- 3. 1,380 (Assabet) 1836

Round 5:

- 1. $(x + 3)^2 + (y + 4)^2 = 7$ (Bartlett)
- 2. $27\frac{7}{9}$ (Assabet)
- 3. (13.5, 9) or $(13\frac{1}{2}, 9)$ (Hudson) (6, 4)

Team Round:

- 1. -3 (QSC)
- 2. 270 (Westboro)
- 3. $\frac{x+1}{2}$ (Auburn)
- 4. (2, 0) (2, -6) or (2, -6) (2, 0) (Shrewsbury)
- 5. $6 + \frac{3\sqrt{21}}{2}$ or 12.874 (St. John's)

- 93
- 6.90 (Burncoat)
- 7. 16 (Notre Dame)
- 8. 40,000 (Bartlett)
- 9. ½ or 0.5 (Tahanto)





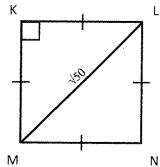
Varsity Meet 3 - January 24, 2018

Round 1: Similarity and Pythagorean Theorem

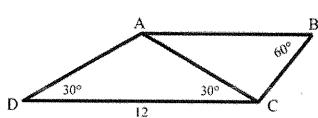
All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Find the perimeter of the square KLMN:



2. ABCD is a trapezoid with AB | | DC. If DC = 12, find AB.



3. Given $\angle C$ is a right angle, E is the midpoint of \overline{AC} , F is the midpoint of \overline{BC} , $\overline{AF} = \sqrt{41}$, and $\overline{BE} = 2\sqrt{26}$, find \overline{AB} .

ANSWERS

(1 pt.) 1.

(2 pt.) 2. _____

(3 pt.) 3. _____



Varsity Meet 3 - January 24, 2018

Round 2: Algebra 1

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

- 1. A + B = 13. A C = -4. Solve for B + C.
- 2. Solve for x:

$$\frac{\frac{2}{3} + \frac{3}{4}x}{4 - \frac{1}{2}x} = \frac{-5}{6}$$

3. Amanda, Bertha, and Charlotte together weigh a total of 500 pounds. If Bertha diets and loses 25% of her weight, then Bertha will weigh the same as a dieting Charlotte who loses 40% of her weight. Together, Bertha and Charlotte will have lost 120 pounds. Find Amanda's weight.

ANSWERS

(1 pt.) 1	
(2 pt.) 2.	
(2 mt) 2	



Varsity Meet 3 – January 24, 2018

Round 3: Functions

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

- 1. The function $h(t) = -13t^2 + 60t + 25$ describes the height, h(t), in feet above the ground of an object t seconds after being thrown straight up from 20 feet above the ground. After how many seconds of being thrown will an object hit the ground?
- 2. $f(x) = x^2$, $g(x) = x \div 3$, h(x) = -x + 2If g(f(x)) = h(f(x)), then find x.
- 3. Consider the quartic equation: $y = x^4 + 5x^3 20x^2 + 20x 96$. Knowing that two of its roots are (2i) and (-2i), determine the other two roots.

ANSWERS

(1 pt.) 1. ______ seconds

(2 pt.) 2. x =

(3 pt.) 3. x = ____ and ____

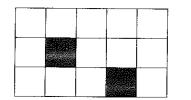
Varsity Meet 3 - January 24, 2018

Round 4: Combinatorics

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

- 1. Suppose a license plate contains two distinct letters followed by three digits with the first digit being nonzero. How many different license plates can be printed?
- 2. In how many ways can the L-shaped figure below (or its mirror image or either of their rotations) be placed in the grid such that it entirely covers 4 white squares in the grid?





3. Jenny, a high school math coach, has three freshmen, two sophomores, six juniors, and three seniors on her team, but she can only use five of them, at most two seniors and at least one from grade 9 or 10. How many unique teams can she form? (Their order on the score sheet is of no consequence.)

<u>ANSWERS</u>

(1 pt.) 1	license plates
(2 pt.) 2.	
(3 pt.) 3	teams

Varsity Meet 3 – January 24, 2018

Round 5: Analytic Geometry

AII	answers	must be	e in	simplest	exact form	in	the	answer	section
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O C	CALCULATOR ALLOWED
1.	Write an equation of a circle in center-radius form with the following center and radius: $c(-3, -4)$; $r = \sqrt{7}$.
2.	A bridge is to be built in the shape of a parabolic arch and is to have a span of 100 feet. The height of the arch, a distance of 40 feet from the center, is to be 10 feet. Find the height of the arch at its center.
3.	Consider the point(s) of intersection of the parabola $y = x^2 - 6x + 9$ and the line $y = x + 3$ What are the coordinates of the center of the circle that passes through the point(s) considered above if the center's x-coordinate is 50% larger than its y-coordinate?

ANSWERS

(1 pt.) 1.	
(2 pt.) 2.	feet
(3 pt.) 3. (.)



Varsity Meet 3 - January 24, 2018

TEAM ROUND

All answers must either ne in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise (2 points each).

APPROVED CALCULATORS ALLOWED

1. The following equation is true for all values of $x \neq 2 \div a$, where a is a constant. What is the value of a?

$$\frac{24x^2 + 25x - 47}{ax - 2} = -8x - 3 - \frac{53}{ax - 2}$$

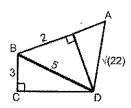
2. How many times does the 1 digit appear exactly twice in the set of 4 digit whole numbers less than 2007?

3. If f(x) = 2x + 5 and g(x) = x - 3, find the inverse of f(g(x)).

4. Find the coordinates of A and B if \overline{AB} is a vertical diameter of the circle with the following equation:

$$x^2 + y^2 - 4x + 6y + 4 = 0$$

5. What is the area of quadrilateral ABCD? (shown below)



6. How many rectangles are in the figure shown below?



7. A rope maker cut a cord into 3 pieces. The first piece was 3 feet long. The second piece is 3 feet plus one-fourth the length of the third piece. The third piece is as long as the first and second pieces put together. How long was the cord before it was cut?

8. If $625 \bullet = 1 \odot$, and $1 \odot = 0.0625 \%$, then how many \bullet 's are in 4 %'s?

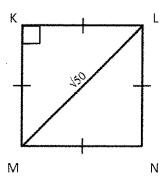
9. Solve for x: $9^{x+2} = 240 + 9^x$



Varsity Meet 3 – SOLUTIONS

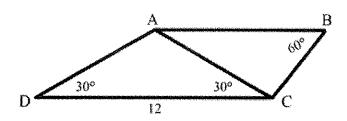
Round 1: Similarity and Pythagorean Theorem

1. Find the perimeter of the square KLMN:

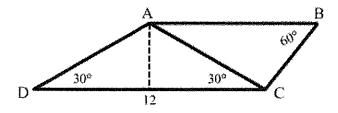


Solution: We know using Pythagorean's theorem that $a^2 + b^2 = c^2$. In this instance, if we focus on one of the two triangles that make up this square, we can deduce that c^2 is equal to 50. Because lengths a and b are the same, we know that $2a^2 = 50$. This means that $a^2 = 25$, and a = 5. The perimeter is the side length multiplied by 4, so the answer is 20.

2. ABCD is a trapezoid with AB | | DC. If DC = 12, find AB.



Solution: First, we'll start by converting ΔACD into two right triangles, as shown below.

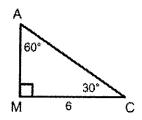


We now see the 30-60-90 triangle, $\triangle AMC$ (where M is the midpoint of \overline{DC} .)



Varsity Meet 3 – SOLUTIONS

Round 1: Similarity and Pythagorean Theorem



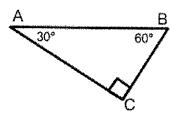
Using the rules of a 30-60-90 triangle, we know $\overline{MC} = \sqrt{3}x$, $\overline{AM} = x$, and $\overline{AC} = 2x$. To solve for x, we write and solve the following equation:

$$6 = \sqrt{3}x$$

$$6\sqrt{3} = 3x$$

$$2\sqrt{3} = x$$

Knowing this, we can now figure out that $\overline{AC} = 4\sqrt{3}$. We already should know that $\triangle ABC$ is another 30-60-90 triangle because lines AB and DC are parallel.



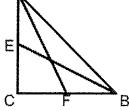
 $\overline{AC} = \sqrt{3}x$, $\overline{BC} = x$, and $\overline{AB} = 2x$. From the information we currently know, we can deduce that $\overline{BC} = 4$, and therefore $\overline{AB} = 8$.



Varsity Meet 3 – SOLUTIONS

Round 1: Similarity and Pythagorean Theorem

3. Given $\angle C$ is a right angle, E is the midpoint of \overline{AC} , F is the midpoint of \overline{BC} , $\overline{AF} = \sqrt{41}$, and $\overline{BE} = 2\sqrt{26}$, find \overline{AB} .



Solution: Let $x = \overline{CF}$ and $y = \overline{CE}$

 ΔACF :

$$(2y)^{2} + x^{2} = (\sqrt{41})^{2}$$
$$4y^{2} + x^{2} = 41$$

 ΔBCE :

$$y^{2} + (2x)^{2} = (2\sqrt{26})^{2}$$
$$y^{2} + 4x^{2} = 104$$

We'll use our system of equations to figure out the value of x and y.

$$y^{2} + 4x^{2} = 104$$

$$y^{2} = -4x^{2} + 104$$

$$4(-4x^{2} + 104) + x^{2} = 41$$

$$-16x^{2} + 416 + x^{2} = 41$$

$$-15x^{2} = -375$$

$$x^{2} = 25$$

$$x = 5$$





Varsity Meet 3 – SOLUTIONS

Round 1: Similarity and Pythagorean Theorem

$$y^{2} + 4(5)^{2} = 104$$
$$y^{2} + 4(5)^{2} = 104$$
$$y^{2} + 100 = 104$$
$$y^{2} = 4$$
$$y = 2$$

Knowing these lengths, we can solve for \overline{AB} .

$$4^{2} + 10^{2} = (AB)^{2}$$
$$116 = (AB)^{2}$$
$$AB = \sqrt{116}$$
$$AB = 2\sqrt{29}$$



Varsity Meet 3 - SOLUTIONS

Round 2: Algebra 1

1.
$$A + B = 13$$
. $A - C = -4$. Solve for $B + C$.

Solution:

$$B = 13 - A$$
 $C = A + 4$
 $B + C = 13 - A + A + 4$
 $B + C = 17$

2. Solve for x:

$$\frac{\frac{2}{3} + \frac{3}{4}x}{4 - \frac{1}{2}x} = \frac{-5}{6}$$

Solution:

$$\frac{\frac{2}{3} + \frac{3}{4}x}{4 - \frac{1}{2}x} = \frac{-5}{6}$$

$$6\left(\frac{2}{3} + \frac{3}{4}x\right) = -5(4 - \frac{1}{2}x)$$

$$4 + \frac{9}{2}x = -20 + \frac{5}{2}x$$

$$8 + 9x = -40 + 5x$$

$$4x = -48$$

$$x = -12$$



Varsity Meet 3 - SOLUTIONS

Round 2: Algebra 1

3. Amanda, Bertha, and Charlotte together weigh a total of 500 pounds. If Bertha diets and loses 25% of her weight, then Bertha will weigh the same as a dieting Charlotte who loses 40% of her weight. Together, Bertha and Charlotte will have lost 120 pounds. Find Amanda's weight.

Solution: Let A = Amanda, B = Bertha, and C = Charlotte.

$$A + B + C = 500$$

$$\frac{3}{4}B = \frac{3}{5}C \Rightarrow B = \frac{4}{5}C$$

$$\frac{1}{4}B + \frac{2}{5}C = 120$$

$$\frac{1}{5}C + \frac{2}{5}C = 120$$

$$\frac{3}{5}C = 120$$

$$C = 200$$

$$B = \frac{4}{5}(200) = 160$$

$$A = 500 - (200 + 160) = 140$$



Varsity Meet 3 - SOLUTIONS

Round 3: Functions

1. The function $h(t) = -13t^2 + 60t + 25$ describes the height, h(t), in feet above the ground of an object t seconds after being thrown straight up from 20 feet above the ground. After how many seconds of being thrown will an object hit the ground?

Solution: To find the time an object hits the ground, we start by setting h(t) equal to 0. We'll be using the quadratic formula: $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$0 = -13t^2 + 60t + 25$$

$$t = \frac{-60 \pm \sqrt{60^2 - 4(-13)(25)}}{2(-13)}$$

$$t = \frac{-60 \pm \sqrt{3600 - (-1300)}}{-26}$$

$$t = \frac{-60 \pm \sqrt{4900}}{-26}$$

$$t = \frac{-60 \pm 70}{-26}$$

We'll have to subtract the numbers in the numerator to get a positive answer, since time cannot be negative.

$$t = \frac{-130}{-26} = 5$$



Varsity Meet 3 – SOLUTIONS

Round 3: Functions

2.
$$f(x) = x^2$$
, $g(x) = x \div 3$, $h(x) = -x + 2$
If $g(f(x)) = h(f(x))$, then find x.

Solution:

$$g(f(x)) = g(x^2) = \frac{x^2}{3}$$

$$h(f(x)) = h(x^2) = -x^2 + 2$$

$$\frac{x^2}{3} = -x^2 + 2$$

$$x^2 = -3x^2 + 6$$

$$4x^2 = 6$$

$$x^2 = \frac{3}{2}$$

$$x = \pm \sqrt{\frac{3}{2}} * \frac{\sqrt{2}}{\sqrt{2}}$$

$$x = \pm \frac{\sqrt{6}}{2}$$

3. Consider the quartic equation: $y = x^4 + 5x^3 - 20x^2 + 20x - 96$. Knowing that two of its roots are (2i) and (-2i), determine the other two roots.

Solution: Given roots (2i) and (-2i),

$$(x - 2i) (x + 2i)$$

$$x^2 - 4i^2$$



Varsity Meet 3 – SOLUTIONS

Round 3: Functions

$$x^2 + 4$$

$$x^{2} + 5x - 24$$

(x + 8) (x - 3)
 $x = -8, 3$

Varsity Meet 3 – SOLUTIONS

Round 4: Combinatorics

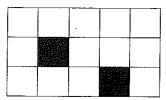
1. Suppose a license plate contains two distinct letters followed by three digits with the first digit being nonzero. How many different license plates can be printed?

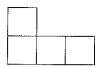
Solution:

- 26 (letters in the alphabet)
- * 25 (letters that aren't the first one thus to make them distinct)
- * 9 (amount of numbers 1-9)
- * 10 (amount of numbers 0-9)
- * 10 (amount of numbers 0-9)

585,000

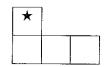
2. In how many ways can the L-shaped figure below (or its mirror image or either of their rotations) be placed in the grid such that it entirely covers 4 white squares in the grid?





Solution: The number in each box is the number of positions with the starred box in that location. Thus, the answer is 17.

0	2	1	2	1
2		3	1	1
0	2	1		1



Varsity Meet 3 - SOLUTIONS

Round 4: Combinatorics

3. Jenny, a high school math coach, has three freshmen, two sophomores, six juniors, and three seniors on her team, but she can only use five of them, at most two seniors and at least one from grade 9 or 10. How many unique teams can she form? (Their order on the score sheet is of no consequence.)

Solution:

F/Soph (5)	J (6)	Sen (3)	
1	3	1	5 * 20 * 3
1	2	2	5 * 15 * 3
2	2	1	10 * 15 * 3
2	1	2	10 * 6 * 3
3	1	1	10 * 6 * 3
3	0	2	10 * 1 * 3
4	0	1	5 * 1 * 3
5	0	0	1 * 1 * 1

300 + 225 + 450 + 180 + 180 + 30 + 15 = 1,380



Varsity Meet 3 – SOLUTIONS

Round 5: Analytic Geometry

1. Write an equation of a circle in center-radius form with the following center and radius: c(-3, -4); $r = \sqrt{7}$.

Solution: Since the center-radius form of a circle is $(x - h)^2 + (y - k)^2 = r^2$, where the center point (h, k), all we need to do is to substitute our information into this form.

$$(x - (-3))^2 + (y - (-4))^2 = (\sqrt{7})^2$$
$$(x + 3)^2 + (y + 4)^2 = 7$$

2. A bridge is to be built in the shape of a parabolic arch and is to have a span of 100 feet. The height of the arch, a distance of 40 feet from the center, is to be 10 feet. Find the height of the arch at its center.

Solution: The format of a parabola is $y = a(x - h)^2 + k$

From this, one can conclude:

1:

$$10 = a(40 - 0)^2 + k$$

$$10 = 1600a + k$$

$$0 = 1600a + k - 10$$

2:

$$0 = a(50 - 0)^2 + k$$

$$0 = 2500a + k$$

Now set the two equations equal to each other.

$$1600a + k - 10 = 2500a + k$$

$$-10 = 900a$$



Varsity Meet 3 - SOLUTIONS

Round 5: Analytic Geometry

$$a=-\frac{1}{90}$$

Then, we substitute a into equation 2:

$$0 = 2500 * -\frac{1}{90} + k$$
$$\frac{2500}{90} = k$$
$$k = \frac{250}{9} = 27\frac{7}{9}$$

3. Consider the point(s) of intersection of the parabola $y = x^2 - 5x + 13$ and the line y = 2x + 7 What are the coordinates of the center of the circle that passes through the point(s) considered above if the center's x-coordinate is 50% larger than its y-coordinate?

Solution:

$$x^{2} - 5x + 13 = 2x + 7$$

$$x^{2} - 7x + 6 = 0$$

$$(x - 6)(x - 1) = 0$$

$$x = 6 \text{ or } 1$$

Now, we substitute this information into either of the equations to find the y-value of each intersection point.

For
$$x = 6$$
, $y = 2(6) + 7 = 19$

For
$$x = 1$$
, $y = 2(1) + 7 = 9$

The intersection points are (6, 19) and (1, 9).

Let the circle center be C(1.5k, k).

Varsity Meet 3 - SOLUTIONS

Round 5: Analytic Geometry

$$CA^{2} = CB^{2}$$

$$(1.5k-6)^{2} + (k-19)^{2} = (1.5k-1)^{2} + (k-9)^{2}$$

$$2.25k^{2} - 18k + 36 + k^{2} - 38k + 361 = 2.25k^{2} - 3k + 1 + k^{2} - 18k + 81$$

$$3.25k^{2} - 56k + 397 = 3.25k^{2} - 21k + 82$$

$$315 = 35k$$

$$9 = k$$

The center is (1.5(9), 9) = (13.5, 9) or $(13\frac{1}{2}, 9)$



Varsity Meet 3 - SOLUTIONS

TEAM ROUND

1. The following equation is true for all values of $x \neq 2 \div a$, where a is a constant. What is the value of a?

$$\frac{24x^2 + 25x - 47}{ax - 2} = -8x - 3 - \frac{53}{ax - 2}$$

Solution: We'll start by multiplying each side by ax - 2.

$$24x^2 + 25x - 47 = (-8x - 3)(ax - 2) - 53$$

$$24x^2 + 25x - 47 = -8ax^2 - 3ax + 16x - 47$$

Since the coefficients of the x^2 term have to be equal on both sides of the equation, -8a = 24, so a = -3.

2. How many times does the *I* digit appear exactly twice in the set of 4 digit whole numbers less than 2007?

Solution: We'll let each _ represent any digit 0-9.

1 1 _ _ = 81 times

 $1_1 = 81 \text{ times}$

 $1_{1} = 81 \text{ times}$

 $2 \ 1 \ 1 = 9 \text{ times}$

 $2.1 _{1} = 9 \text{ times}$

2 - 1 1 = 9 times

When we add all those up, we get 270 times.

3. If f(x) = 2x + 5 and g(x) = x - 3, find the inverse of f(g(x)).



Varsity Meet 3 – SOLUTIONS TEAM ROUND

Solution:

$$f(g(x)) = 2(x-3) + 5 = 2x-6+5=2x-1$$

To find the inverse, we need to set our equation equal to y and then "switch" x and y.

$$y = 2x - 1$$

$$x = 2y - 1$$

$$x + 1 = 2y$$

$$y = \frac{x + 1}{2}$$

4. Find the coordinates of A + B if \overline{AB} is a vertical diameter of the circle with the following equation:

$$x^2 + y^2 - 4x + 6y + 4 = 0$$

Solution:

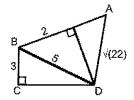
$$x^{2} + y^{2} - 4x + 6y + 4 = 0$$

$$x^{2} - 4x + 4 + y^{2} + 6y + 9 = -4 + 13$$

$$(x - 2)^{2} + (y + 3)^{2} = 9$$

We can now infer the center point of the circle is C(2, -3) and the radius is 3. This means that point A and B must be at $(2, -3\pm3)$, which is (2, 0) and (2, -6).

5. What is the area of quadrilateral ABCD? (shown below)





Varsity Meet 3 – SOLUTIONS TEAM ROUND

Solution: We know $\overline{CD} = 4$ because ΔBCD is a 3-4-5 right triangle. We'll call the point between B and A 'X'.

$$\overline{XD} = \sqrt{5^2 - 2^2} = \sqrt{21}$$

Now, we'll solve for the length of \overline{XA} .

$$\overline{XA} = \sqrt{22 - 21} = \sqrt{1} = 1$$

$$Area = \frac{1}{2}(3)(4) + \frac{1}{2}(3)(\sqrt{21}) = 6 + \frac{3\sqrt{21}}{2}$$

6. How many rectangles are in the figure shown below?



Solution: Below is a chart showing how many of each sized rectangle is in the figure.

1 x 1 = 15	2 x 1 = 10	$3 \times 1 = 5$	
$1 \times 2 = 12$	$2 \times 2 = 8$	$3 \times 2 = 4$	
$1 \times 3 = 9$	$2 \times 3 = 6$	$3 \times 3 = 3$	
$1 \times 4 = 6$	$2 \times 4 = 4$	$3 \times 4 = 2$	
$1 \times 5 = 3$	$2 \times 5 = 5$	$3 \times 5 = 1$	

Adding these gives the sum of 90.

7. A rope maker cut a cord into 3 pieces. The first piece was 3 feet long. The second piece is 3 feet plus one-fourth the length of the third piece. The third piece is as long as the first and second pieces put together. How long was the cord before it was cut?

Solution: Let x = piece 1, y = piece 2, and z = piece 3.



Varsity Meet 3 – SOLUTIONS TEAM ROUND

$$x = 3$$

$$y = 3 + \frac{1}{4}z$$

$$z = x + y = 3 + y$$

$$y = 3 + \frac{1}{4}(3 + y)$$

$$4y = 12 + 3 + y$$

$$3y = 15$$

$$y = 5$$

$$z = 3 + 5 = 8$$

The cord is 3 + 5 + 8 = 16 feet.

8. If $625 \stackrel{\blacktriangle}{\bullet} = 1 \stackrel{\frown}{\odot}$, and $1 \stackrel{\frown}{\odot} = 0.0625 \stackrel{\maltese}{\circ}$, then how many $\stackrel{\blacktriangle}{\bullet}$'s are in 4 $\stackrel{\maltese}{\circ}$'s?

Solution: If
$$1 \odot = 0.0625$$
 then 1 = $16 \odot$

So
$$4 \% = 16(4) \odot = 64 \odot$$

9. Solve for x:
$$9^{x+2} = 240 + 9^x$$

Solution:

$$9^{x+2} = 240 + 9^x$$

$$9^{x+2} - 9^x = 240$$

$$9^x(81 - 1) = 240$$



Varsity Meet 3 – SOLUTIONS TEAM ROUND

$$9^x(80) = 240$$

$$9^x = 3$$

$$x = \frac{1}{2}$$
 or 0.5